Evaluating asset pricing models in a Fama-French framework

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This version: June 2008.

Abstract

In this work we propose a new methodology to compare different stochastic discount factor (SDF) proxies based on relevant market information. The starting point is the work of Fama and French, which evidenced that the asset returns of the U.S. economy could be explained by relative factors linked to characteristics of the firms. In this sense, we construct a Monte Carlo simulation to generate a set of returns perfectly compatible with the Fama and French factors and, then, investigate the performance of different SDF proxies. We use some goodness-of-fit statistics and the Hansen Jagannathan distance as a formal criterion to compare asset-pricing models. An empirical application of our setup is also provided, revealing that the novel non-parametric estimator proposed by Araujo et al. (2006) exhibit, in general, the best performance among several traditional SDF proxies.

Keywords: Asset Pricing, Fama and French model, Stochastic Discount Factor, Hansen-Jagannathan distance.

JEL Codes: G12, C15, C22.

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1 Introduction

In this work, we propose a new methodology to compare different stochastic discount factor or pricing kernel proxies. In asset pricing theory, one of the major interests for empirical researchers is oriented by testing whether a particular asset pricing model is (indeed) supported by the data. In addition, a formal procedure to compare the performance of competing asset pricing models is also of great importance in empirical applications. In both cases, it is of utmost relevance to establish an objective measure of model misspecification. The most useful measure is the well-known Hansen and Jagannathan (1997) distance (or simply HJ-distance), which has been used both as a model diagnostic tool and as a formal criterion to compare asset pricing models. This type of comparison has been employed in many recent papers.

As argued by Hansen and Richard (1987), observable implications of candidate models of asset markets are conveniently summarized in terms of their implied stochastic discount factors. As a result, some recent studies of the asset pricing literature have been focused on proposing an estimator for the SDF and also on comparing competing pricing models in terms of the SDF model. For instance, see Lettau and Ludvigson (2001b), Chen and Ludvigson (2008), Araujo, Issler and Fernandes (2006).

A different route to investigate and compare asset pricing models has also been suggested in the literature. The main idea is to assume a data generation process (DGP) for a set of asset returns, based on some assumptions about the asset prices and, then, create a controlled framework, which is used to evaluate and compare the asset pricing models. In this sense, Fernandes and Vieira (2006) study, through Monte Carlo simulations, the performance of different SDF estimatives at different environments. Firstly, the authors consider that all asset prices follow a geometric Brownian motion. In this case, one should expect that a SDF proxy based on a geometric Brownian motion assumption would have a better performance, in comparison to an asset pricing model that does not assume this hypothesis. The authors also study competing asset pricing models in a stationary Ornstein-Uhlenbeck process as done in Vasicek (1977). However, a critical issue of this procedure is that the best asset pricing model inside these particular environments (i.e., when the asset prices are

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1 We use the term “stochastic discount factor” as a label for a state-contingent discount factor.

2 For instance, by using the HJ-distance, Campbell and Cochrane (2000) explain why the CAPM and its extensions better approximate asset pricing models than the standard consumption based model; Jagannathan and Wang (2002) compare the SDF method with Beta method in estimating a risk premium; Dittmar (2002) uses the HJ-distance to estimate the nonlinear pricing kernels in which the risk factor is endogenously determined and preferences restrict the definition of the pricing kernel. Other examples in the literature include Jagannathan, Kubota and Takehara (1998), Farnsworth, Ferson, Jackson, and Todd (2002), Lettau and Ludvigson (2001a) and Chen and Ludvigson (2008).
supposed to follow a geometric Brownian motion or a stationary Ornstein-Uhlenbeck process), might not be a good model in the real world. In other words, the best estimator for each controlled framework might not necessarily exhibit the same performance for observed stock market prices of a real economy.

In this paper, we use the controlled approach of Fernandes and Vieira (2006), but instead of generating the asset returns from an ad-hoc assumption about the DGP of returns, we use related market information from the real economy. Our starting point is the work of Fama and French, which evidenced that asset returns of the U.S. economy could be explained by relative factors linked to characteristics of the firms\(^3\). Based on the Fama and French factors, we firstly construct a Monte Carlo simulation to generate a set of returns that is perfectly compatible with these factors. The next step is to create a framework to compare the competing asset pricing models. To do so, we consider two sets of returns: The first sample is used to estimate the different SDF proxies, whereas the remaining sample is used to analyze the out-of-sample performance of each asset pricing model. Notice that this procedure could also be adopted to compare models by using real data, but with some limitations since the DGP is unknown.\(^4\)

Finally, because our approach enables us to construct a data generation process of the SDF provided by the Fama and French specification, it is possible to compare competing proxies through some goodness-of-fit statistics. In addition, it is relevant to test if a set of SDF candidates satisfy the law of one price, such that \(1 = E_t(m_{t+1}R_{t+1})\), where \(m_{t+1}\) is referred to the investigated stochastic discount factor. Thus, we say that a SDF correctly "prices" the assets if this equation is (in fact) satisfied. In this sense, we test the previous restriction by evaluating the HJ-distance of each SDF candidate model. As shown by Hansen and Jagannathan, the HJ-distance \(\delta = \min_{m \in \mathcal{M}} \|y - m\|\), defined in the \(L^2\) space, is the distance of the SDF model \(y\) to a family of SDFs, \(m \in \mathcal{M}\), that correctly price the assets. In other interpretation, Hansen and Jagannathan show that the HJ-distance is the pricing error for the portfolio that is most mispriced by the underlying model. In this sense, even though the investigated SDF models are misspecified, in practical terms, we are interested in those models with the lowest HJ-distance.

Therefore, we propose a new methodology to compare different stochastic discount factors by using the market information in a Fama and French (1992, 1993) environment. The main

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\(^3\)Fama and French (1993, 1995) argue that a three-factor model is successful because it proxies for unobserved common risk in portfolio returns.

\(^4\)Although we do not directly use market returns data in this paper, we are able to compare different SDFs by using important market information provided by the Fama-French factors.
objective here is not to propose a DGP process of actual market returns, but to provide a controlled environment that allows us to properly compare and evaluate different SDF proxies. This work follows the idea of Farnsworth et al. (2002), which study different SDFs by constructing artificial mutual funds using real stock returns from the CRSP data.

To illustrate our methodology, we present an empirical application, in which several SDF models are compared: a) The novel non-parametric estimator of Araujo, Issler and Fernandes (2006); b) The Brownian motion pricing model of Brandt, Cochrane and Saint-Clara (2006); c) The projection on the payoff space estimator of Hansen and Jagannathan (1991); and d) the (traditional) unconditional linear CAPM. Overall, the estimator proposed by Araujo, Issler and Fernandes (2006) exhibits the best performance in comparison to the other asset pricing models.

This work is organized as follows: Section 2 presents the Fama and French model and describes the Monte Carlo simulation strategy; Section 3 presents the results of the empirical application; and Section 4 shows the main conclusions.

2 The stochastic discount factor and the Fama and French model

A general framework to asset pricing is well described in Harrison and Kreps (1979), Hansen and Richard (1987) and Hansen and Jagannathan (1991), associated to the stochastic discount factor (SDF), which relies on the pricing equation:

\[ p_t = E_t (m_{t+1}x_{i,t+1}) , \]  
\[ (1) \]

where \( E_t(\cdot) \) denotes the conditional expectation given the information available at time \( t \), \( p_t \) is the asset price, \( m_{t+1} \) the stochastic discount factor, \( x_{i,t+1} \) the asset payoff of the \( i \)-th asset in \( t+1 \). This pricing equation means that the market value today of an uncertain payoff tomorrow is represented by the payoff multiplied by the discount factor, also taking into account different states of nature by using the underlying probabilities.

The stochastic discount factor model provides a general framework for pricing assets. As documented by Cochrane (2001), asset pricing can basically be summarized by two equations:

\[ p_t = E_t [m_{t+1}x_{t+1}] , \]  
\[ (2) \]
\[ m_{t+1} = f (\text{data, parameters}) . \]  
\[ (3) \]

This way, one can conveniently separate the task of specifying economic assumptions from the task of deciding which kind of empirical representation to pursue. Based on a given model
represented by the function \( f(\cdot) \), we next show how the pricing equation (2) can lead to different predictions stated in terms of returns.

For instance, in the Consumption-based Capital Asset Pricing Model (CCAPM) context, the first-order conditions of the consumption-based model, summarized by the well-known Euler equation:

\[
p_t = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right] x_{t+1}.
\]

The specification of \( m_{t+1} \) corresponds to the intertemporal marginal rate of substitution. Hence,

\[
m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)},
\]

where \( \beta \) is the discount factor for the future, \( c_t \) is consumption and \( u(\cdot) \) is a given utility function. The pricing equation (2) mainly illustrates the fact that consumers (optimally) equate marginal rates of substitution to prices.

2.1 Fama and French framework

Fama and French (1992) show that, besides the market risk, there are other important factors that help explain the average return in the stock market. This evidence has been demonstrated in several works for different stock markets (see Gaunt (2004) and Griffin (2005) for a good review). Although there is not a clear link between these factors and the economic theory (e.g., CAPM model), these evidences show that some additional factors might (quite well) help to understand the dynamics of the average return.

These factors are known as the size and the book-to-market equity and represent special features about firms. Fama and French (1992) argue that size and book-to-market equity are indeed related to economic fundamentals. Although they appear to be "ad hoc variables" in an average stock returns regression, these authors justify them as expected and natural proxies for common risk factors in stock returns.

The factors

\( (i) \) The SMB (Small Minus Big) factor is constructed to measure the size premium. In fact, it is designed to track the additional return that investors have historically received by investing in stocks of companies with relatively small market capitalization. A positive SMB in a given month indicates that small cap stocks have outperformed the large cap stocks in that month. On the other hand, a negative SMB suggests that large caps have outperformed.

\( (ii) \) The HML (High Minus Low) factor is constructed to measure the premium-value provided to investors for investing in companies with high book-to-market values (essentially, the value placed on the company by accountants as a ratio relative to its market value, commonly expressed as B/M). A positive HML in a given month suggests that “value stocks” (i.e., high B/M) have
outperformed the “growth stocks” (low B/M) in that month, whereas a negative HML indicates that growth stocks have outperformed.\footnote{Notice that, in respect to SMB, small companies logically are expected to be more sensitive to many risk factors, as a result of their relatively undiversified nature, and also their reduced ability to absorb negative financial events. On the other hand, the HML factor suggests higher risk exposure for typical value stocks in comparison to growth stocks.}

\textit{(iii) The Market factor} is the market excess return in comparison to the risk-free rate. For instance, we proxy the excess return on the market \((R_M - R_f)\), in the U.S economy, by the value-weighted portfolio of all stocks listed on the New York Stock Exchange (NYSE), the American Stock Exchange (AMEX), and NASDAQ stocks (from CRSP) minus the one-month Treasury Bill rate.

\textit{The Model}

Fama and French (1993, 1996) propose a three-factor model for expected returns (see also Fama and French (2004) for a good survey).

\begin{equation}
E(R_{it}) - R_{ft} = \beta_{im} [E(R_{Mt}) - R_{ft}] + \beta_{is} E(SMB_t) + \beta_{ih} E(HML_t), \quad i \in \{1, \ldots, N\}, \quad (4)
\end{equation}

where the betas \(\beta_{im}, \beta_{is}\) and \(\beta_{ih}\) are slopes in the multiple regression (4). Hence, one implication of the expected return equation of the three-factor model is that the intercept in the time-series regression (5) is zero for all assets \(i\):

\begin{equation}
R_{it} - R_{ft} = \beta_{im} (R_{Mt} - R_{ft}) + \beta_{is} SMB_t + \beta_{ih} HML_t + \varepsilon_{it}. \quad (5)
\end{equation}

Using this criterion, Fama and French (1993, 1996) find that the model captures much of the variation in the average return for portfolios formed on size, book-to-market equity and other price ratios.
Expected return - beta representation

The Fama and French approach is (in fact) a multifactor model, that can be seen as an expected-beta\(^6\) representation of linear factor pricing models of the form:

\[
E(R_i) = \gamma + \beta_{im}\lambda_m + \beta_{is}\lambda_s + \beta_{ih}\lambda_h + \alpha_i, \quad i \in \{1, ..., N\}.
\]  

(6)

If we run this cross sectional regression of average returns on betas, one can estimate the parameters (\(\gamma, \lambda_m, \lambda_s, \lambda_h\)). Notice that \(\gamma\) is the intercept and \(\lambda_m, \lambda_s\) and \(\lambda_h\) the slope in this cross-sectional relation. In addition, the \(\beta_{im}, \beta_{is}\) and \(\beta_{ih}\) are the unconditional sensitivities of the \(i\)-th asset to the factors\(^7\). Moreover, \(\beta_{ij}\), for some \(j \in \{m, s, h\}\), can be interpreted as the amount of risk exposure of asset \(i\) to factor \(j\), and \(\lambda_j\) as the price of such risk exposure. Hence, the betas are defined as the coefficients in a multiple regression of returns on factors:

\[
R_{it} - R_{ft} = \beta_{im}R_{Mt}^e + \beta_{is}SMB_t + \beta_{ih}HML_t + \varepsilon_{it}, \quad t \in \{1, ..., T\},
\]  

(7)

where \(R_{Mt}^e = (R_{Mt} - R_{ft})\). Following the equivalence between this beta-pricing model and the linear model for the discount factor \(M\), in an unconditional setting (see Cochrane, 2001), we can estimate \(M\) as:

\[
M = a + b'f,
\]  

(8)

where \(f = [R_M^e, SMB, HML]^\prime\), and the relations between \(\lambda\), \(\gamma\), and \(a\) and \(b\), are given by:

\[
a = \frac{1}{\gamma} \quad \text{and} \quad b = -\gamma [\text{cov} (f f')]^{-1} \lambda.
\]  

(9)

### 2.2 Evaluating the performance of competing models

In the asset pricing literature, some measures are suggested to compare competing asset pricing models. The most famous measure is the Hansen and Jagannathan distance. However, as long as the data generation process (DGP) is known in each specification of the Fama and French model, it is also possible to use some simple sample statistics. This route is adopted by Fernandes and Vieira (2006) to compare the relative performance among SDF proxies, based on goodness-of-fit statistics to assess whether the estimators correctly proxy the DGP. In addition, we use the Hansen and Jagannathan distance to test for model misspecification and to compare the performance of different asset pricing models.

\(^6\)The main objective of the beta model is to explain the variation in terms of average returns across assets.

\(^7\)An unconditional time-series approach is used here. The conditional approaches to test for international pricing models include those by Ferson & Harvey (1994, 1999) and Chan, Karolyi and Stulz (1992).
The Hansen-Jagannathan (1997) distance measure is a summary of the mean pricing errors across a group of assets. The HJ measure may also be interpreted as the distance between the SDF candidate and one that would correctly price the primitive assets. The pricing error can be written by \( \alpha_t = E_t (m_{t+1} R_{i,t+1}) - 1 \). Notice, in particular, that \( \alpha_t \) depends on the considered SDF, and the SDF is not unique (unless markets are complete). Thus, different SDF proxies can produce similar HJ measures.

**Goodness-of-fit statistics**

We use three goodness-of-fit statistics. The first one (\( \overline{MSE}_s \)) is merely a standardized version of the mean squared error of the SDF proxies. The second one (\( \hat{\gamma}_s \)) compares the sample correlation between the actual and estimated stochastic discount factors. Finally, the last one (\( \hat{p}_s \)) verifies whether the SDF proxies satisfy the unconditional version of the pricing equation; \( 1 = E_t (M_{t+1} R_{i,t+1}) \). Notice the later measure is different from the HJ-distance.

Let \( M_t \) be the stochastic discount factor generated by the Fama and French specification (DGP), and \( \hat{M}_t^s \) the SDF proxy provided by model \( s \) in a family \( S \) of asset pricing models. The standardized mean squared error is thus computed as:

\[
\overline{MSE}_s = \frac{\sum_{t=1}^{T} (\hat{M}_t^s - M_t)^2}{\sum_{t=1}^{T} M_t^2}, \quad \text{for } s \in S. \tag{10}
\]

The sample correlation between the actual and estimated SDF is given by:

\[
\hat{\gamma}_s = \text{corr}(\hat{M}_t^s, M_t), \quad \text{for } s \in S, \tag{11}
\]

and the pricing equation statistic is given by:

\[
\hat{p}_s = \frac{1}{T \times N} \sum_{i=1}^{N} \sum_{t=1}^{T} (\hat{M}_t^s R_{i,t} - 1)^2, \quad \text{for } s \in S. \tag{12}
\]
2.3 Constructing the Fama and French environment

Based on the assumption that $R_{Mt}$, $SMB_t$ and $HML_t$ are known variables, we can reproduce a Fama and French environment following the three factors of the Fama and French model:

$$R_{i,t} - R_{ft} = \beta_{im} (R_{Mt} - R_{ft}) + \beta_{is} SMB_t + \beta_{ih} HML_t + \varepsilon_{it}. \quad (13)$$

The simulated asset returns are generated using the equation (13). This way, we propose the following steps of a Monte Carlo simulation:

1) Firstly, calibrate each parameter $\beta^k_{ij}$, for $j \in \{m, s, h\}$ and $i \in \{1, ..., N\}$ according to previous estimations of Fama and French (1992, 1993). Therefore, we will generate for each $j$ a $N$-dimensional vector of asset returns.

2) By considering $\beta^k_{ij}$ created in step 1 for some $i \in \{1, ..., N\}$ and using the known factors $R_{Mt}$, $SMB_t$ and $HML_t$, we generate a returns vector along the time dimension, through equation (7). The iid shock $\varepsilon_{it}$ is assumed to be a white noise with zero mean and constant variance.

3) Repeating step 2 for each $i \in \{1, ..., N\}$, we create the matrix $R^k$ of asset returns, in which rows are formed by different returns and columns represent the time dimension.

4) Evaluate the mean of $R^k$ across each row to generate a cross-section vector. Now, it is possible to estimate the parameters $\gamma^k$ and $\lambda^k$ through equation (6).

5) Estimate parameters $a^k$ and $b^k$ from the equivalence relation shown in equation (9). Finally, the stochastic discount factor can be estimated by using equation (8).

6) Repeat steps 1 to 5 for an amount of $K$ replications in order to construct the Monte Carlo simulation.

7) We use an amount of $N$ assets, obtained in steps 1 to 6, to compare the competing asset pricing models.

7.a) Split the set of $N$ assets into two groups (with the same number of time series observations for each group). Firstly, consider an amount of $\tilde{N} < N$ assets to estimate the SDF candidates (henceforth, this first group of assets will be denominated "in-sample"). On the other hand, the remaining $N - \tilde{N}$ assets will be used to compare the SDF proxies based on the described goodness-of-fit statistics (i.e., "out-of-sample" investigation).
7.b) Every SDF proxy will be compared with the correct SDF provided by the Fama and French model, which is computed by using the out-of-sample \((N - \hat{N})\) assets information. The HJ-distance of each \(y\) candidate model is thus compared with the Hansen and Jagannathan SDF proxy, which is also estimated from the out-of-sample information\(^8\).

7.c) Finally, because our approach enables us to construct a data generation process of the SDF provided by the Fama and French specification, it is possible to compare competing proxies through the goodness-of-fit statistics described in the previous section.

Notice that a similar procedure could be employed to compare SDF models by using real data, despite the fact that the DGP is unknown in this case. The HJ-distance could also be used, as commonly suggested in literature.

### 3 Empirical Application

In this section, we present an empirical exercise of our proposed framework to compare alternative models of the stochastic discount factor, commonly discussed in the literature.

**A. Hansen and Jagannathan (Primitive-Efficient Stochastic Discount Factors)**

Hansen and Jagannathan (1991) proposed an identification of \(M_{t+1}\) by the projection of \(M_{t+1}\) onto the space of payoffs, which makes it straightforward to express \(M^*_{t+1}\), the mimicking portfolio, only as a function of observables. They describe a general framework to asset pricing, associated to the stochastic discount factor, which relies on the pricing equation:

\[
1 = E_t [M_{t+1}R_{t+1}].
\]

(14)

By considering the stochastic discount factor strictly positive and some additional hypotheses in the previous pricing equation (14), these authors show that the mimicked discount factor \(M^*_{t+1}\) has a direct relation to the minimal conditional variance portfolio. Moreover, they exploit the fact that it is always possible to project the SDF onto the space of payoffs, which makes it straightforward to express the mimicking portfolio as a function of only observable values:

\[
M^*_{t+1} = t_N^' [E_t (R_{t+1} R_{t+1}')]^{-1} R_{t+1},
\]

(15)

where \(t_N\) is a \(N \times 1\) vector of ones, and \(R_{t+1}\) is a \(N \times 1\) vector stacking all asset returns. Equation (15) delivers a nonparametric estimate of the SDF that is solely a function of asset returns. However,

\(^8\)By construction, the H&J estimator of the pricing kernel correctly satisfies the equation: \(E_t (m_{t+1} R_{t+1}) = 1.\)
one disadvantage of this procedure consists in evaluating the conditional moment \( E_t (R_{t+1} R'_{t+1}) \).

Notice that, as long as the number of assets increases, it becomes more difficult to estimate \( M_{t+1} \) through (15). In the limit case, i.e. \( N \to \infty \), the matrix \( E_t (R_{t+1} R'_{t+1}) \) will be of infinite order. Even for a finite but large number of assets, possibly there will be singularities in that matrix, since the correlation between some assets may be very close to unity. Therefore, equation (15) reflects the fact that \( M_{t+1} \) is a linear function of a conditional minimum-variance efficient portfolio. We also refer to this estimative as a Primitive-Efficient SDF.

B. Brandt, Cochrane and Santa-Clara (2006)

Brandt, Cochrane and Santa-Clara (2006) consider that the asset prices follow a geometric Brownian motion (GBM). Such hypothesis is defined by the following partial differential equation:

\[
\frac{dP}{P} = \left( R^f + \mu \right) dt + \Sigma^{1/2} dB,
\]

where, \( \frac{dP}{P} = \left( \frac{dP_1}{P_1}, ..., \frac{dP_N}{P_N} \right)' \), \( \mu = (\mu_1, ..., \mu_n)' \), \( \Sigma \) is a \( N \times N \) positive definite matrix, \( P_i \) is the price of the asset \( i \), \( \mu \) the risk premium vector, \( R^f \) the risk free rate, and \( B \) a standard GBM of dimension \( N \). Using Itô theorem, it is possible to show that:

\[
R_{t+\Delta t}^i = \frac{P_{t+\Delta t}^i}{P_t^i} = e^{\left( R^f + \mu - \frac{1}{2} \Sigma \right) \Delta t + \sqrt{\Delta t} \left( \Sigma^{1/2} \right)' Z_t},
\]

where \( Z_t \) is a vector of \( N \) independent variables with Gaussian distribution. Therefore, the SDF proposed by these authors is calculated as

\[
M_{t+\Delta t} = e^{-\left( R^f + \frac{1}{2} \Sigma \right) \Delta t - \sqrt{\Delta t} \left( \Sigma^{1/2} \right)' Z_t}.
\]

Thus, Brandt, Cochrane and Santa-Clara (2006) suggest the following SDF estimator:

\[
\hat{M}_t = e^{-\left( \hat{R}^f + \frac{1}{2} \hat{\Sigma} \right) \Delta t - \sqrt{\Delta t} \left( \hat{\Sigma}^{1/2} \right)' \hat{Z}_t}.
\]

where, \( \hat{\mu}, \hat{R} \) and \( \hat{\Sigma} \) are estimated by:

\[
\hat{\mu} = \frac{\bar{R} - R^f}{\Delta t},
\]

\[
\hat{\Sigma} = \frac{1}{\Delta t T} \sum_{t=1}^{T} (R_t - \bar{R}) (R_t - \bar{R})',
\]

such that, \( R_t = (R_{t1}, ..., R_{tN})' \) and \( \bar{R} = \frac{1}{T} \sum_{t=1}^{T} R_t \).
C. Araujo, Issler and Fernandes (2006)

A novel estimator for the stochastic discount factor (within a panel data context) is proposed by Araujo, Issler and Fernandes (2006). This setting is slightly more general than the GBM setup put forth by Brandt, Cochrane and Santa-Clara (2006). In fact, this estimator assumes that, for every asset $i \in \{1, ..., N\}$, $M_{t+1}R_{i, t+1}$ is conditionally homoskedastic and has a lognormal distribution. In addition, under asset pricing equation (14) and some mild additional conditions, they show that a consistent estimator for $M_t$ is given by:

$$\tilde{M}_t = \left( \frac{\bar{R}^G}{\bar{R}^A (\bar{R}^G)^{\frac{A}{G}}} \right),$$

where $\bar{R}^A_t = \frac{1}{N} \sum_{i=1}^{N} R_{i,t}$ and $\bar{R}^G_t = \prod_{i=1}^{N} R_{i,t}^{-\frac{1}{N}}$ are respectively the cross-sectional arithmetic and geometric average of all gross returns. Therefore, this nonparametric estimator depends exclusively on appropriate averages of asset returns that can easily be implemented.

D. Capital Asset Pricing Model - CAPM

Assuming the unconditional CAPM, the SDF is a linear function of market returns calculated as: $m_{t+1} = a + bR_{w,t+1}$, where $R_{w,t+1}$ is the gross return on the market portfolio of all assets. For instance, in the U.S. economy, in order to implement the static CAPM, for practical purposes, it is commonly assumed that the return on the value-weighted portfolio of all stocks listed on NYSE, AMEX, and NASDAQ is a reasonable proxy for the return on the market portfolio of all assets of the U.S. economy.

3.1 Monte Carlo design

Our strategy in the numerical simulation is aimed to simulate the real econometricians behavior, when they compute a SDF model. In this sense, we construct a DGP from the Fama and French setup, by considering $N = 36$ as our set of primitive assets.

In order to estimate the stochastic discount factors, we firstly specify an adequate setup for the Fama and French environment. To do so, we study the SDF proxies under the specifications represented by equations (13). The first step is to consider the factors $(R_{Mt} - R_{ft}), SMB_t$ and $HML_t$ of the U.S. economy, which are extracted from the Kenneth R. French website.\(^9\)

\(^9\)More information about data can be found in:
http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
Next, we calibrate the parameters $\beta_{im}$, $\beta_{is}$ and $\beta_{ih}$ according to previous estimations of Fama and French (1992, 1993) and estimate the parameters $(\gamma, \lambda_m, \lambda_s, \lambda_h)$ from the cross-sectional regression (6), observing their significance through the $F$-statistic or the $t$-statistic for individual parameters. We set $N = 36$ as our set of primitive assets which will be divided into two groups of assets. The first group contains $\tilde{N} = 18$ assets that will be used for the in-sample estimation. The second group has $(N - \tilde{N}) = 18$ assets, which are used in the out-of-sample analysis. We also consider, for each generated asset $i$, three sample sizes $T = \{200; 300; 400\}$.

This way, we estimate the stochastic discount factors for the three-factor model of Fama and French, and repeat the mentioned procedure for an amount of $K = 1,000$ replications. Some descriptive statistics of the generated SDFs are presented in appendix. Finally, the evaluation of the SDF proxies is conducted and the Monte Carlo results are summarized by three goodness-of-fit statistics (besides the HJ-distance), which are averaged across all replications. The robustness of the results is analyzed through different sample sizes.

In each replication of the Monte Carlo simulation, the following SDF proxies are estimated:

- $\widehat{M}^a_i$: Stochastic discount factor of Araujo, Issler and Fernandes (2006);
- $\widehat{M}^b_i$: Stochastic discount factor of Brandt, Cochrane and Santa-Clara (2006);
- $\widehat{M}^c_i$: Stochastic discount factor of Hansen and Jagannathan (1991);
- $\widehat{M}^d_i$: Stochastic discount factor of the unconditional CAPM;
- $M_i$: Stochastic discount factor implied by the Fama and French setup (DGP).

### 3.2 Results

In Figure 1, the estimates of the SDF proxies are shown for one replication of the Monte Carlo simulation, with a sample size $T = 200$. A simple graphical investigation reveals that the estimatives of Brandt et al. (2006), $\widehat{M}^b_i$, and of Araujo et al. (2006), $\widehat{M}^a_i$, are respectively the most and less volatile, which is a result confirmed by the descriptive statistics of Table 1 (in appendix).
Notes: a) Figure 1 shows one replication out of the total amount of 1,000 replications.

b) We adopt $N=18$ assets and $T=200$ observations.

Regarding the performance of the SDF proxies, Table 1 reports the evaluation statistics provided by the Monte Carlo simulation. Notice that results are robust to sample size. Firstly, note that the mean square error satisfies the following inequalities (for $T = 200$ and 300): $\hat{MSE}_a < \hat{MSE}_b < \hat{MSE}_c < \hat{MSE}_d$. The Araujo et al. (2006) SDF proxy shows quite a good performance, followed by the Brandt, Cochrane and Santa-Clara (2006) estimator, whereas the Hansen and Jagannathan (1991) and the CAPM model seem to exhibit the worst performance.

In respect to the correlation of the true SDF with the considered SDF proxies, we have obtained the following ranking order ($T = 200$ and 400): $\hat{M}_t^a > \hat{M}_t^b > \hat{M}_t^c > \hat{M}_t^d$. This implies that the Araujo et al. (2006) proxy (in general) best tracks the dynamic path of the true SDF. On the other hand, the CAPM model exhibits again the worst performance (with a negative correlation in all sample sizes!)

Regarding the $\hat{p}_a$ goodness-of-fit statistic, we have obtained the following result (in all sample sizes): $\hat{p}_a < \hat{p}_d < \hat{p}_b < \hat{p}_c$, revealing that the CAPM is the second-best model according to this criterion, probably due to the unconditional nature of this statistic.
Finally, in respect to the HJ distance\textsuperscript{10} (which for the corrected-specified model should be as close as possible to zero), notice that the Brandt, Cochrane and Santa-Clara (2006) and Hansen and Jagannathan (1991) estimators are the best ones in all sample sizes, followed by the Araujo et al. (2006) proxy. The CAPM estimator exhibits the worst performance.

Putting all together, the numerical results show that (in general) the SDF proxy of Araujo et al. (2006) has the best performance, possibly due to its nonparametric nature and quite weak assumptions. On the other hand, the CAPM model shows a negative correlation with the true SDF, revealing its weakness in tracking the real dynamic of the true SDF. This result is because the linear CAPM only use one factor of the correct three factor specifications of Fama-French. In addition, the good response in the $\hat{p}_s$ criterion is since the SDF proxy is few volatile.

Moreover, the Hansen and Jagannathan (1991) and Brandt et al. (2006) SDFs do not have a good performance in the Fama and French setup. This result for the Hansen and Jagannathan proxy is possibly due to the fact that the estimation of the inverse of the matrix $E_t \left(R_{t+1} R_{t+1}'\right)$ may exhibit near singularities for a finite but large number of assets. In order to estimate this matrix, we imposed $T >> N$, since this proxy is only feasible if $T > N(N + 1)/2$. On the other hand, in the Brandt et al. (2006) estimator, the geometric Brownian motion hypothesis is possibly a strong assumption for the considered set of asset returns.

4 Conclusions

In the present work, we propose a new methodology to compare different stochastic discount factor (SDF) proxies based on relevant market information. The starting point is the work of Fama and French, which evidenced that the asset returns of the U.S. economy could be explained by relative factors linked to characteristics of the firms. Thus, we construct a "Fama-French world" through a Monte Carlo simulation to generate a set of returns that is perfectly compatible with those factors.

In this sense, we present an empirical application in which returns time series are produced according to the Fama and French environment, based on factors such as the market portfolio return, size and book-to-market equity of the U.S. economy. Then, several stochastic discount factor models are compared: a) Araujo, Issler and Fernandes (2006); b) Brandt, Cochrane and Saint-Clara (2006); c) Hansen and Jagannathan (1991); and d) CAPM. The results indicate that the SDF proxy suggested by Araujo et al. (2006) dominates some traditional SDF estimators.

\textsuperscript{10}We compute the HJ distance based on the MatLab codes of Mike Cliff, available at: http://mcliff.cob.vt.edu/
This controlled framework allows us to use simple sample statistics to compare the SDF candidates with the true SDF implied by the Fama and French DGP. In addition, we use the Hansen and Jagannathan distance as a formal measure of model misspecification. As a natural extension of this work, the proposed methodology could easily be adapted to compare asset pricing models based on real asset returns data.

Furthermore, several important topics remain for future research. For instance, one might adopt a finite sample correction in the HJ-distance, in order to properly test for the equality between the SDF proxies and the implied Fama and French SDF (e.g., Ren & Shimotsu (2006) or Kan & Robotti (2008)). The details of this route remain an issue to be further explored.

References


Appendix

Table 1 - Descriptive statistics of the SDF proxies

<table>
<thead>
<tr>
<th>sample size = 200</th>
<th>Araujo</th>
<th>Saint Clara</th>
<th>H &amp; J</th>
<th>CAPM</th>
<th>DGP (FF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0,9836</td>
<td>0,9178</td>
<td>0,9911</td>
<td>0,9721</td>
<td>0,9969</td>
</tr>
<tr>
<td>Median</td>
<td>0,9816</td>
<td>0,8361</td>
<td>0,9959</td>
<td>0,9925</td>
<td>0,9998</td>
</tr>
<tr>
<td>Maximum</td>
<td>1,2884</td>
<td>2,9999</td>
<td>2,0772</td>
<td>1,4243</td>
<td>2,1053</td>
</tr>
<tr>
<td>Minimum</td>
<td>0,6611</td>
<td>0,1643</td>
<td>-0,3800</td>
<td>0,2015</td>
<td>-0,5192</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>0,0688</td>
<td>0,4217</td>
<td>0,4035</td>
<td>0,1851</td>
<td>0,3364</td>
</tr>
<tr>
<td>Skewness</td>
<td>0,2395</td>
<td>1,5188</td>
<td>-0,2095</td>
<td>-0,6558</td>
<td>-0,5526</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>8,5455</td>
<td>7,4555</td>
<td>3,5370</td>
<td>4,1835</td>
<td>6,0852</td>
</tr>
<tr>
<td>Freq. Jarque-Bera</td>
<td>0,0000</td>
<td>0,0000</td>
<td>0,6470</td>
<td>0,0000</td>
<td>0,0000</td>
</tr>
</tbody>
</table>

| sample size = 300 |
|-------------------|--------|-------------|-------|------|---------|
| Mean              | 0,9863 | 0,9193      | 0,9906| 0,9754| 0,9960  |
| Median            | 0,9802 | 0,8550      | 0,9950| 0,9900| 0,9881  |
| Maximum           | 1,5155 | 3,0337      | 1,9893| 1,4169| 2,1840  |
| Minimum           | 0,6618 | 0,2076      | -0,1875| 0,0993| -0,3064 |
| Std. Dev          | 0,0701 | 0,3689      | 0,3471| 0,1582| 0,3062  |
| Skewness          | 1,6362 | 1,5898      | -0,1485| -0,8925| -0,2483 |
| Kurtosis          | 16,5960| 8,7703      | 3,3921| 6,2824| 5,3345  |
| Freq. Jarque-Bera | 0,0000 | 0,0000      | 0,6800| 0,0000| 0,0000  |

| sample size = 400 |
|-------------------|--------|-------------|-------|------|---------|
| Mean              | 0,9820 | 0,9175      | 0,9899| 0,9738| 0,9952  |
| Median            | 0,9775 | 0,8666      | 0,9978| 0,9783| 1,0039  |
| Maximum           | 1,5138 | 3,0631      | 1,9813| 1,2156| 2,1623  |
| Minimum           | 0,6615 | 0,1472      | -0,5119| 0,5767| -0,6715 |
| Std. Dev          | 0,0738 | 0,3401      | 0,3294| 0,0759| 0,3033  |
| Skewness          | 1,5307 | 1,6841      | -0,4395| -0,4045| -0,9122 |
| Kurtosis          | 14,4988| 10,0147     | 4,7585| 5,4933| 9,2944  |
| Freq. Jarque-Bera | 0,0000 | 0,0000      | 0,0370| 0,0000| 0,0000  |

Notes: The descriptive statistics are averaged across the K=1,000 replications based on the sample sizes T={200,300,400}. For instance, for T=200 the Jarque-Bera statistic indicates the frequency of rejection of the normality hypothesis across the 1,000 replications (based on a 5% significance level). In this case, in Table 1, T=200, for the Hansen & Jagannathan proxy, the statistic Freq. Jarque-Bera is equal to 0.6470, which means that in 64.70% of the replications the normality hypothesis is rejected at a 5% significance level. The number of "in-sample" and "out-of-sample" assets is N=18.
Table 2 - Fama and French model with \((R_{Mt} - R_{ft})\), \(SMB\) and \(HML\) factors

<table>
<thead>
<tr>
<th></th>
<th>Araujo (SDFa)</th>
<th>Saint Clara (SDFb)</th>
<th>H &amp; J (SDFc)</th>
<th>CAPM (SDFd)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample size = 200</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSEa</td>
<td>0.0768</td>
<td>0.1027</td>
<td>0.1157</td>
<td>0.1855</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>0.0220</td>
<td>0.0332</td>
<td>0.0328</td>
<td>0.0547</td>
</tr>
<tr>
<td>Mean corr_a</td>
<td>0.7284</td>
<td>0.6646</td>
<td>0.5611</td>
<td>-0.4223</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>0.0720</td>
<td>0.0672</td>
<td>0.0832</td>
<td>0.1171</td>
</tr>
<tr>
<td>Mean p_a</td>
<td>0.0239</td>
<td>1.5367</td>
<td>1.5531</td>
<td>0.5544</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>0.0036</td>
<td>0.3583</td>
<td>0.3869</td>
<td>0.2793</td>
</tr>
<tr>
<td>HJ dist. SDFa</td>
<td>0.3784</td>
<td>0.3200</td>
<td>0.3203</td>
<td>0.4591</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>0.0789</td>
<td>0.0754</td>
<td>0.0773</td>
<td>0.0733</td>
</tr>
<tr>
<td>Sample size = 300</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSEa</td>
<td>0.0652</td>
<td>0.0705</td>
<td>0.0853</td>
<td>0.1524</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>0.0163</td>
<td>0.0226</td>
<td>0.0231</td>
<td>0.0407</td>
</tr>
<tr>
<td>Mean corr_a</td>
<td>0.7192</td>
<td>0.5860</td>
<td>-0.4433</td>
<td></td>
</tr>
<tr>
<td>Std. Dev</td>
<td>0.0720</td>
<td>0.0603</td>
<td>0.0797</td>
<td>0.0828</td>
</tr>
<tr>
<td>Mean p_a</td>
<td>0.0217</td>
<td>1.1227</td>
<td>1.1267</td>
<td>0.4209</td>
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<tr>
<td>Std. Dev</td>
<td>0.0031</td>
<td>0.2331</td>
<td>0.2572</td>
<td>0.1893</td>
</tr>
<tr>
<td>HJ dist. SDFa</td>
<td>0.3211</td>
<td>0.2538</td>
<td>0.2555</td>
<td>0.3959</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>0.0636</td>
<td>0.0598</td>
<td>0.0622</td>
<td>0.0596</td>
</tr>
<tr>
<td>Sample size = 400</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSEa</td>
<td>0.0584</td>
<td>0.0569</td>
<td>0.0667</td>
<td>0.1121</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>0.0129</td>
<td>0.0149</td>
<td>0.0164</td>
<td>0.0339</td>
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<tr>
<td>Mean corr_a</td>
<td>0.7984</td>
<td>0.7476</td>
<td>0.6527</td>
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<tr>
<td>Std. Dev</td>
<td>0.0507</td>
<td>0.0501</td>
<td>0.0685</td>
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<td>Mean p_a</td>
<td>0.0207</td>
<td>0.8762</td>
<td>0.9940</td>
<td>0.2051</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>0.0028</td>
<td>0.1572</td>
<td>0.1910</td>
<td>0.1528</td>
</tr>
<tr>
<td>HJ dist. SDFa</td>
<td>0.2940</td>
<td>0.2251</td>
<td>0.2225</td>
<td>0.3540</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>0.0546</td>
<td>0.0512</td>
<td>0.0532</td>
<td>0.0524</td>
</tr>
</tbody>
</table>

Notes: The underlined numbers show the best SDF within each goodness-of-fit statistic. All results are averaged across the 1,000 replications. All statistics are computed for the "out-of-sample" set of assets (N=18).

The DGP is constructed from parameters \(\beta_{1m}\in[0.2;1.0]\), \(\beta_{1s}\in[-0.7;2.3]\), \(\beta_{1h}\in[-0.7;2.3]\).